Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 190 (2006) 270–286

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

The ABC of hyper recursions

Amparo Gil[a], Javier Segura[a,*], Nico M. Temme[b]

[a]*Departamento de Matemáticas, Estadística y Computación, Univ. Cantabria, 39005-Santander, Spain*
[b]*CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands*

Received 30 September 2004

Dedicated to Roderick Wong on the occasion of his 60th birthday

Abstract

Each member of the family of Gauss hypergeometric functions

$$f_n = {}_2F_1(a+\varepsilon_1 n, b+\varepsilon_2 n; c+\varepsilon_3 n; z),$$

where a, b, c and z do not depend on n, and $\varepsilon_j = 0, \pm 1$ (not all ε_j equal to zero) satisfies a second order linear difference equation of the form

$$A_n f_{n-1} + B_n f_n + C_n f_{n+1} = 0.$$

Because of symmetry relations and functional relations for the Gauss functions, the set of 26 cases (for different ε_j values) can be reduced to a set of 5 basic forms of difference equations. In this paper the coefficients A_n, B_n and C_n of these basic forms are given. In addition, domains in the complex z-plane are given where a pair of minimal and dominant solutions of the difference equation have to be identified. The determination of such a pair asks for a detailed study of the asymptotic properties of the Gauss functions f_n for large values of n, and of other Gauss functions outside this group. This will be done in a later paper.

MSC: 33C05; 39A11; 65D20

Keywords: Gauss hypergeometric functions; Recursion relations; Difference equations; Stability of recursion relations; Numerical evaluation of special functions

* Corresponding author.
E-mail addresses: amparo.gil@unican.es (A. Gil), javier.segura@unican.es (J. Segura), nicot@cwi.nl (N.M. Temme).

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.01.041

1. Introduction

The Gauss hypergeometric functions

$$f_n = {}_2F_1\left(\begin{matrix} a+\varepsilon_1 n, b+\varepsilon_2 n \\ c+\varepsilon_3 n \end{matrix}; z\right) \tag{1.1}$$

where ε_j are integers, a, b, c and z do not depend on n, and

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n, \quad |z|<1 \tag{1.2}$$

satisfy a *second order linear difference equation* (also called a *three-term recurrence relation*) of the form

$$A_n f_{n-1} + B_n f_n + C_n f_{n+1} = 0. \tag{1.3}$$

For example, we have

$$(c-a)\,{}_2F_1\left(\begin{matrix} a-1, b \\ c \end{matrix}; z\right) + (2a+-c-z(a-b))\,{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) + a(z-1)\,{}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; z\right) = 0, \tag{1.4}$$

in which we can replace a with $a+n$. Other examples are given in [1, p. 558].

In this paper we consider the 26 recursion relations with respect to n for the cases

$$\varepsilon_j = 0, \pm 1, \quad j = 1, 2, 3, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \neq 0, \tag{1.5}$$

and by using symmetry relations and functional relations for the Gauss functions we assign a set of 5 basic forms from which the remaining 21 cases can be obtained.

A solution f_n of the recurrence relation (1.3) is said to be *minimal* if there exists a linearly independent solution g_n, of the same recurrence relation such that $f_n/g_n \to 0$ as $n \to \infty$. In that case g_n is called a *dominant* solution. When a recurrence admits a minimal solution (unique up to a constant factor), this solution should be included in any numerically satisfactory pair of solutions of the recurrence. Given a solution of the recurrence relation, it is crucial to know the character of the solution (minimal, dominant or none of them) in order to apply the recurrence relation in a numerically stable way. Indeed, if f_n is minimal as $n \to +\infty$, forward recurrence (increasing n) is an ill conditioned process because small initial errors will generally dominate the recursive solution by introducing an initially small component of a dominant solution; backward recurrence is well conditioned in this case. The opposite situation takes place for dominant solutions.

For each basic form we give the coefficients A_n, B_n and C_n, and after computing limits of the ratios $\beta = \lim_{n\to\infty}(A_n/C_n)$ and $\alpha = \lim_{n\to\infty}(B_n/C_n)$ we determine the zeros t_1 and t_2 of the characteristic polynomial $t^2+\alpha t+\beta$, and we give curves in the z-plane where $|t_1|=|t_2|$. These curves enclose domains where a pair $\{f_n, g_n\}$ of minimal and dominant solutions of the difference equation has to be identified. For each basic form, and for each domain in the z-plane defined by the boundary curves belonging to that form, we give a number of candidates of minimal and dominant solutions. The selection of a suitable pair $\{f_n, g_n\}$ of minimal and dominant solutions can be done after a detailed study of the asymptotic behaviour of the Gauss functions ${}_2F_1(a+\varepsilon_1 n, b+\varepsilon_2 n; c+\varepsilon_3 n; z)$ for large values of n, with z a fixed complex number, and of other Gauss functions outside this group. This will be done in a second paper. We conclude with a numerical example that shows the efficiency of using a particular difference equation.

2. Basic recursion relations and their solutions

There are 26 recursion relations for these functions for all choices of ε_j. However, we can use several functional relations in order to reduce our study to few basic recursions equations. First, we have the symmetry relation

$$ {}_2F_1\left({a, b \atop c}; z\right) = {}_2F_1\left({b, a \atop c}; z\right). \tag{2.1} $$

In addition, the following relations can be used:

$$ {}_2F_1\left({a, b \atop c}; z\right) = (1-z)^{-a}\,{}_2F_1\left({a, c-b \atop c}; \frac{z}{z-1}\right), \tag{2.2} $$

$$ {}_2F_1\left({a, b \atop c}; z\right) = (1-z)^{-b}\,{}_2F_1\left({c-a, b \atop c}; \frac{z}{z-1}\right), \tag{2.3} $$

$$ {}_2F_1\left({a, b \atop c}; z\right) = (1-z)^{c-a-b}\,{}_2F_1\left({c-a, c-b \atop c}; z\right). \tag{2.4} $$

See [1, p. 559]. These relations show that a great number of the 26 cases are equivalent or follow from each other. In fact we can find 5 basic forms that need to be studied, while the remaining cases follow from the relations in (2.1)–(2.4).

When $c = 0, -1, -2, \ldots$ the Gauss hypergeometric function in (1.2) is not defined. Also, when a or b assume non-positive integer values, the series in (1.2) terminates. In the following we will not distinguish about these special cases and the general cases, and we assume that all representations of the functions to be given are well defined, and we will not specify that special values of the parameters should be excluded in the results.

In Table 1 we give an overview of all possible recursions, and indicate the five basic forms. Observe that we take recursion in positive direction equivalent with recursion in negative direction, however we will need to distinguish between both directions when studying the asymptotic behavior of the solutions; see also Section 7. Apart from the notation in Table 1, we will also use the notation $(\mathrm{sign}(\varepsilon_1)\,\mathrm{sign}(\varepsilon_2)\,\mathrm{sign}(\varepsilon_3))$ when $|\varepsilon_j| \leqslant 1$.

For example, the case $k = 4$ will be also denoted by $(+\,0\,+)$. This means that we consider the recursion relation for the function

$$ f_n = {}_2F_1\left({a+n, b \atop c+n}; z\right). \tag{2.5} $$

By using (2.3) we can write this as

$$ f_n = (1-z)^{-b}\,{}_2F_1\left({c-a, b \atop c+n}; \zeta\right), \quad \zeta = \frac{z}{z-1}. \tag{2.6} $$

Table 1
Only five basic forms remain

k	ε_1	ε_2	ε_3	Type	Comments
1	1	1	1	$\equiv 13$	Use (2.4)
2	1	1	0	Basic form	
3	1	1	−1	Basic form	
4	1	0	1	$\equiv 13$	Use (2.3)
5	1	0	0	Basic form	
6	1	0	−1	Basic form	
7	1	−1	1	$\equiv 16$	Use (2.3)
8	1	−1	0	$\equiv 2$	Use (2.2)
9	1	−1	−1	$\equiv 6$	Use (2.2)
10	0	1	1	$\equiv 13$	Use (2.2)
11	0	1	0	$\equiv 5$	Use (2.1)
12	0	1	−1	$\equiv 6$	Use (2.1)
13	0	0	1	Basic form	
14	0	0	0		Void
15	0	0	−1	$\equiv 13$	Change sign in 13
16	0	−1	1	$\equiv 12$	Change signs in 12
17	0	−1	0	$\equiv 11$	Use (2.2)
18	0	−1	−1	$\equiv 15$	Use (2.2)
19	−1	1	1	$\equiv 7$	Use (2.1)
20	−1	1	0	$\equiv 8$	Use(2.1)
21	−1	1	−1	$\equiv 9$	Use (2.1)
22	−1	0	1	$\equiv 16$	Use (2.1)
23	−1	0	0	$\equiv 17$	Use (2.1)
24	−1	0	−1	$\equiv 18$	Use (2.1)
25	−1	−1	1	$\equiv 3$	Change signs in 3
26	−1	−1	0	$\equiv 2$	Use (2.4)
27	−1	−1	−1	$\equiv 15$	Use (2.4)

If we have a pair $\{F_n, G_n\}$ of minimal and dominant solutions of recursion relation related with the basic form $k = 13$, we can use relation (2.3) to obtain a pair $\{f_n, g_n\}$ for the recursion relation related with $k = 4$.

In the next section we give other functional relations between the Gauss hypergeometric functions, in which always one Gauss function is expressed in terms of two other ones. These relations are important for finding the second solution of the linear difference equations, and in many cases they are important for obtaining asymptotic information. They will not be used for reducing further the 5 basic forms.

3. Selecting a second solution

Once we have reduced the number of basic recursions to be studied to 5, we will give for each basic form the coefficients A_n, B_n and C_n of the recursion relation (1.3) and we have to study the character of

f_n as a solution of the corresponding recurrence relation. For this, we will need to find a second solution of the relation which forms a satisfactory pair of solutions together with f_n (i.e. a pair which includes the minimal solution when it exists).

This second solution can be chosen from several connection formulas between the hypergeometric functions. For example, the functions

$$ {}_2F_1\left({a,b \atop c};z\right) \quad \text{and} \quad z^{1-c}{}_2F_1\left({a-c+1,b-c+1 \atop 2-c};z\right) \tag{3.1}$$

both satisfy the differential equation (see [7, p. 112])

$$ z(1-z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + ((c-(a+b+1)z)\frac{\mathrm{d}w}{\mathrm{d}z} - abw = 0. \tag{3.2}$$

Both functions in (3.1) also satisfy the same difference equation, when a suitable normalization for the second function is chosen. To obtain this normalization we use the relation

$$ \begin{aligned} {}_2F_1\left({a,b \atop c};z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1\left({a,b \atop a+b-c+1};1-z\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1\left({c-a,c-b \atop c-a-b+1};1-z\right), \end{aligned} \tag{3.3}$$

where $|\mathrm{phase}(1-z)| < \pi$ (see [1, Eq. 15.3.6] or [7, p. 113]). By replacing z by $1-z$ and $a+b-c+1$ by c we can write this in the form

$$ \begin{aligned} {}_2F_1\left({a,b \atop c};z\right) &= P\,{}_2F_1\left({a,b \atop a+b-c+1};1-z\right) \\ &\quad - Qz^{1-c}{}_2F_1\left({a-c+1,b-c+1 \atop 2-c};z\right), \end{aligned} \tag{3.4}$$

where $|\mathrm{phase}(z)| < \pi$ and

$$ P = \frac{\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(a+b-c+1)\Gamma(1-c)}, \quad Q = \frac{\Gamma(c-1)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(a)\Gamma(b)\Gamma(1-c)}. \tag{3.5}$$

For example, for the recursion in (1.4) we may consider as a second solution the second term in (3.4), replacing a by $a+n$, and deleting the factors that do not depend on n. This gives the two solutions of (1.4)

$$ \begin{aligned} f_n &= {}_2F_1\left({a+n,b \atop c};z\right), \\ g_n &= \frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}{}_2F_1\left({a+n-c+1,b-c+1 \atop 2-c};z\right). \end{aligned} \tag{3.6}$$

The principal questions now are:

1. Are these two functions linear independent solutions of (1.4)?
2. Do they constitute a numerically satisfactory pair? That is, is one member dominant with respect to the other one for large n?
3. If these two questions are answered affirmatively, in which domain(s) of the z-plane hold these properties?

Considering the fact that both f_n and g_n are of type $(+\,0\,0)$, and that in this choice of g_n no other factors occur that significantly influence the asymptotic behaviour of g_n, we cannot expect that the first two questions are answered affirmatively. Taking the first term in (3.4), again deleting in P the nonrelevant gamma functions, we obtain the pair

$$f_n = {}_2F_1\left({a+n, b \atop c}; z\right),$$

$$g_n = \frac{\Gamma(a+n+1-c)}{\Gamma(a+n+b-c+1)}\, {}_2F_1\left({a+n, b \atop a+n+b-c+1}; 1-z\right), \tag{3.7}$$

in which the Gauss function for g_n is now of type $(+\,0\,+)$, which seems to be a better choice. By using (2.3), however, we obtain

$$g_n = \frac{\Gamma(a+n+1-c)}{\Gamma(a+n+b-c+1)} z^{-b}\, {}_2F_1\left({b-c+1, b \atop a+n+b-c+1}; \frac{z-1}{z},\right) \tag{3.8}$$

where $|\mathrm{phase}(z)| < \pi$. The asymptotics of this g_n easily follows from the power series in (1.2). The Gauss function is $1 + \mathcal{O}(1/n)$ for large values n. Asymptotic properties of f_n are needed to conclude if the pair in (3.7) constitute a satisfactory pair, and if so, in which domain(s) of the z-plane.

Another connection formula to be used is

$$\begin{aligned} {}_2F_1\left({a, b \atop c}; z\right) &= \frac{\Gamma(1-a)\Gamma(b-c+1)}{\Gamma(1-c)\Gamma(b-a+1)}(-z)^{a-c}(1-z)^{c-a-b}\, {}_2F_1\left({1-a, c-a \atop b-a+1}; \frac{1}{z}\right) \\ &\quad - \frac{\Gamma(c-1)\Gamma(b-c+1)\Gamma(1-a)}{\Gamma(b)\Gamma(c-a)\Gamma(1-c)}(-z)^{1-c}(1-z)^{c-a-b}\, {}_2F_1\left({1-a, 1-b \atop 2-c}; z\right), \end{aligned} \tag{3.9}$$

where $|\mathrm{phase}(-z)| < \pi$. This relation follows from (see [1, Eq. 15.3.7])

$$\begin{aligned} {}_2F_1\left({a, b \atop c}; z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\, {}_2F_1\left({a, 1-c+a \atop 1-b+a}; \frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\, {}_2F_1\left({b, 1-c+b \atop 1-a+b}; \frac{1}{z}\right), \end{aligned} \tag{3.10}$$

after we change $z \to 1/z$ and use the result (2.4).

By taking the second term of (3.9), deleting the factors that are constant in the recursion, we obtain as a second solution for the $(+\,0\,0)$ case the function

$$g_n = (1-z)^{-n}\frac{\Gamma(1-a-n)}{\Gamma(c-a-n)}\,{}_2F_1\left({1-a-n, 1-b \atop 2-c}; z\right). \tag{3.11}$$

The gamma functions of the form $\Gamma(\alpha - n)$, with integer n, can be replaced by using the relation (see [7, p. 74])

$$\Gamma(\alpha - n) = \frac{(-1)^n \pi}{\sin(\pi\alpha)\Gamma(n+1-\alpha)}. \tag{3.12}$$

This gives

$$g_n = (1-z)^{-n}\frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}\,{}_2F_1\left({1-a-n, 1-b \atop 2-c}; z\right) \tag{3.13}$$

and, by using (2.3), it follows that we can also take

$$g_n = (1-z)^{-n}\frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}\,{}_2F_1\left({1-c+a+n, 1-b \atop 2-c}; \frac{z}{z-1},\right) \tag{3.14}$$

in which the final Gauss function is of type $(+\,0\,0)$, just as f_n in (3.6). But this Gauss function has a different z-argument, and this g_n has a factor $(1-z)^{-n}$ in its representation. Again, more asymptotic properties of f_n are needed to conclude if this g_n is a proper companion to form a satisfactory pair with f_n.

Other connection forms to be used for obtaining the second solution are (see [1, p. 559] or [7, p. 113])

$$\begin{aligned}{}_2F_1\left({a, b \atop c}; z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-z)^{-a}\,{}_2F_1\left({a, c-b \atop 1-b+a}; \frac{1}{1-z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left({b, c-a \atop 1-a+b}; \frac{1}{1-z}\right),\end{aligned} \tag{3.15}$$

where $|\mathrm{phase}(1-z)| < \pi$, and

$$\begin{aligned}{}_2F_1\left({a, b \atop c}; z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z^{-a}\,{}_2F_1\left({a, 1-c+a \atop a+b-c+1}; \frac{z-1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{a-c}(1-z)^{c-a-b}\,{}_2F_1\left({c-a, 1-a \atop c-a-b+1}; \frac{z-1}{z}\right),\end{aligned} \tag{3.16}$$

where $|\mathrm{phase}(1-z)| < \pi$ and $|\mathrm{phase}(z)| < \pi$.

From the connection formulas given in this section new forms can be derived, for example by replacing $1/(1-z)$ by z in (3.15) or $(z-1)/z$ by z in (3.16) and by using (2.2)–(2.4). Each term in the connection formulas satisfies the hypergeometric differential equation (3.2), and all these manipulations give Kummer's 24 solutions of the differential equation. For the complete collection, see [1, p. 563]. All these 24 functions, properly normalized with fractions of gamma functions, can also be used to construct (for a certain domain of the z-plane) a pair of minimal and dominant solutions of a given basic recurrence relation.

4. Coefficients for the basic forms

We give the coefficients A_n, B_n and C_n of the basic recursions. In each case we give the solution f_n and a possible candidate g_n for constructing a pair of minimal and dominant solutions. However, neither f_n nor g_n may play a role in identifying minimal and dominant solutions. In a later paper we give full details about this. In the next section we will obtain the regions of existence of minimal solutions by means of Perron's theorem.

4.1. Basic form $k = 2$

The $(+\,+\,0)$ recursion relation reads

$$A^{(2)}(a+n, b+n)y_{n-1} + B^{(2)}(a+n, b+n)y_n + C^{(2)}(a+n, b+n)y_{n+1} = 0, \tag{4.1}$$

where

$$\begin{aligned} A^{(2)}(a, b) &= (c-a)(c-b)(c-a-b-1),\\ B^{(2)}(a, b) &= (c-a-b)\{c(a+b-c)+c-2ab\\ &\qquad + z[(a+b)(c-a-b)+2ab+1-c]\},\\ C^{(2)}(a, b) &= ab(c-a-b+1)(1-z)^2, \end{aligned} \tag{4.2}$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left(\begin{matrix} a+n, b+n \\ c \end{matrix}; z\right),\\ g_n &= \frac{\Gamma(a+n+1-c)\Gamma(b+n+1-c)}{\Gamma(a+b+2n-c+1)}\,{}_2F_1\left(\begin{matrix} a+n, b+n \\ a+b+2n-c+1 \end{matrix}; 1-z\right). \end{aligned} \tag{4.3}$$

The second solution is taken from the first term in Eq. (3.4).

As we will later show, when $z \leqslant 0$ the recurrence has no minimal solutions, whereas in compact domains that do not contain points of $(-\infty, 0]$, a pair of minimal and dominant solutions exists.

This case has applications for Jacobi polynomials. We have

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix} -n, -\beta-n \\ \alpha+1 \end{matrix}; z\right), \quad z = \frac{x-1}{x+1}. \tag{4.4}$$

A representation with $+n$ at the a and b places follows from applying (2.4). In the interval of orthogonality $-1 \leqslant x \leqslant 1$, we have $z \leqslant 0$, and if $x \in [-1, 1]$ the recursion relation of the Jacobi polynomials can be used for computing these functions in forward direction. Only the usual rounding errors should be taken into account.

4.2. Basic form $k = 3$

The $(+\,+\,-)$ recursion relation reads

$$\begin{aligned} &A^{(3)}(a+n, b+n, c-n)y_{n-1} + B^{(3)}(a+n, b+n, c-n)y_n\\ &\quad + C^{(3)}(a+n, b+n, c-n)y_{n+1} = 0, \end{aligned} \tag{4.5}$$

where

$$
\begin{aligned}
&A^{(3)}(a,b,c) = -(a-c)(a-c-1)(b-1-c)(b-c)zU,\\
&B^{(3)}(a,b,c) = c[c_1U + c_2V + c_3UV],\\
&c_1 = (1-z)(b-c)(b-1)[a-1+z(b-c-1)],\\
&c_2 = b(b+1-c)(1-z)(a+z(b-c+2)),\\
&c_3 = c-2b-(a-b)z,\\
&C^{(3)}(a,b,c) = abc(c-1)(1-z)^3V,\\
&U = z(a+b-c+1)(a+b-c+2)+ab(1-z),\\
&V = (1-z)(1-a-b+ab)+z(a+b-c-1)(a+b-c-2),
\end{aligned}
\tag{4.6}
$$

with solutions given by

$$
\begin{aligned}
f_n &= {}_2F_1\left(\begin{matrix} a+n, b+n \\ c-n \end{matrix}; z\right),\\
g_n &= \frac{\Gamma(a+1-c+2n)\Gamma(b+1-c+2n)}{\Gamma(a+b-c+1+3n)\Gamma(1-c+n)}\,{}_2F_1\left(\begin{matrix} a+n, b+n \\ a+b-c+1+3n \end{matrix}; 1-z\right),
\end{aligned}
\tag{4.7}
$$

where we have used the first term in Eq. (3.4).

4.3. Basic form $k=5$

The $(+\,0\,0)$ recursion relation reads

$$A^{(5)}(a+n)y_{n-1} + B^{(5)}(a+n)y_n + C^{(5)}(a+n)y_{n+1} = 0, \tag{4.8}$$

where

$$
\begin{aligned}
&A^{(5)}(a) = c-a,\\
&B^{(5)}(a) = 2a-c-(a-b)z\\
&C^{(5)} = a(z-1),
\end{aligned}
\tag{4.9}
$$

with solutions given by

$$
\begin{aligned}
f_n &= {}_2F_1\left(\begin{matrix} a+n, b \\ c \end{matrix}; z\right),\\
g_n &= (1-z)^{-n}\frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}\,{}_2F_1\left(\begin{matrix} 1-a-n, 1-b \\ 2-c \end{matrix}; z\right),
\end{aligned}
\tag{4.10}
$$

where we have used the first term in the right-hand side of (3.9).

4.4. Basic form $k=6$

The $(+\,0\,-)$ recursion relation reads

$$A^{(6)}(a+n,c-n)y_{n-1}+B^{(6)}(a+n,c-n)y_n+C^{(6)}(a+n,c-n)y_{n+1}=0, \tag{4.11}$$

where

$$\begin{aligned} A^{(6)}(a,c) &= z(a-c)(a-c-1)(b-c)[a+z(b+1-c)],\\ B^{(6)}(a,c) &= c[a(a-1)(c-1)+a(a-1)(a+3b-4c+2)z\\ &\quad +(b-c)(b+1-c)(4a-c-1)z^2-(a-b)(b-c)(b+1-c)z^3],\\ C^{(6)}(a,c) &= -ac(c-1)[a-1+z(b-c)](1-z)^2, \end{aligned} \tag{4.12}$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left({a+n,b \atop c-n};z\right),\\ g_n &= \frac{\Gamma(a+1-c+2n)\Gamma(b+1-c+n)}{\Gamma(a+b-c+1+2n)\Gamma(1-c+n)}\,{}_2F_1\left({a+n,b \atop a+b-c+1+2n};1-z\right), \end{aligned} \tag{4.13}$$

where g_n is selected from the first term of Eq. (3.4).

4.5. Basic form $k=13$

The $(0\,0\,+)$ recursion relation reads

$$A^{(13)}(c+n)y_{n-1}+B^{(13)}(c+n)y_n+C^{(13)}(c+n)y_{n+1}=0, \tag{4.14}$$

where

$$\begin{aligned} A^{(13)}(c) &= c(c-1)(z-1),\\ B^{(13)}(c) &= c[c-1-(2c-a-b-1)z],\\ C^{(13)}(c) &= (c-a)(c-b)z, \end{aligned} \tag{4.15}$$

with solutions given by

$$\begin{aligned} f_n &= {}_2F_1\left({a,b \atop c+n};z\right),\\ g_n &= \frac{(-1)^n(1-z)^n\Gamma(c+n)}{\Gamma(c-a-b+1+n)}\,{}_2F_1\left({c-a+n,c-b+n \atop c-a-b+1+n};1-z\right), \end{aligned} \tag{4.16}$$

where we have used the second term in the right-hand side of (3.3).

5. Domains for minimal and dominant solutions

Perron's theorem (see [9, Appendix B]) gives in the case of finite limits the following results.

Given a difference (1.3) we define

$$\alpha := \lim_{n\to\infty} \frac{B_n}{C_n}, \quad \beta := \lim_{n\to\infty} \frac{A_n}{C_n}. \tag{5.1}$$

Let t_1 and t_2 denote the zeros of the characteristic polynomial $t^2 + \alpha t + \beta = 0$. Then, if $|t_1| \neq |t_2|$ the difference equation has two linear independent solutions f_n and g_n with the properties

$$\frac{f_{n+1}}{f_n} \sim t_1, \quad \frac{g_{n+1}}{g_n} \sim t_2. \tag{5.2}$$

If $|t_1| = |t_2|$, then

$$\limsup_{n\to\infty} |y_n|^{1/n} = |t_1| \tag{5.3}$$

for any nontrivial solution y_n of (1.3).

In the following subsections, we give for the five basic forms the domains in the z-plane where $|t_1| \neq |t_2|$. In these domains there is a true distinction between the two solutions of (1.3). If $|t_1| > |t_2|$ then the solution f_n that satisfies the relation in (5.2) is a maximal solution and g_n is the minimal solution. On the curves where $|t_1| = |t_2|$ the two solutions are neither dominant nor minimal, and recursion in forward or backward direction is not unstable.

For all five basic forms the ratios A_n/C_n and B_n/C_n of the difference equation (1.3) tend to finite limits as $|n| \to \infty$. Interestingly, in all cases these limits are functions of z, and they are not dependent on the parameters a, b or c.

5.1. *The domains for basic form $k = 2$*

The limits α and β of (4.1) are

$$\alpha = -\frac{2(z+1)}{(1-z)^2}, \quad \beta = \frac{1}{(1-z)^2}. \tag{5.4}$$

The zeros of the characteristic polynomial are

$$t_1 = \frac{1}{\left(1-\sqrt{z}\right)^2}, \quad t_2 = \frac{1}{\left(1+\sqrt{z}\right)^2}. \tag{5.5}$$

The equation $|t_1| = |t_2|$ holds when $z \leqslant 0$, otherwise $|t_1| > |t_2|$.

5.2. *The domains for basic form $k = 3$*

The limits α and β of (4.5) are

$$\alpha = \frac{8z^2 + 20z - 1}{(1-z)^3}, \quad \beta = -\frac{16z}{(1-z)^3}. \tag{5.6}$$

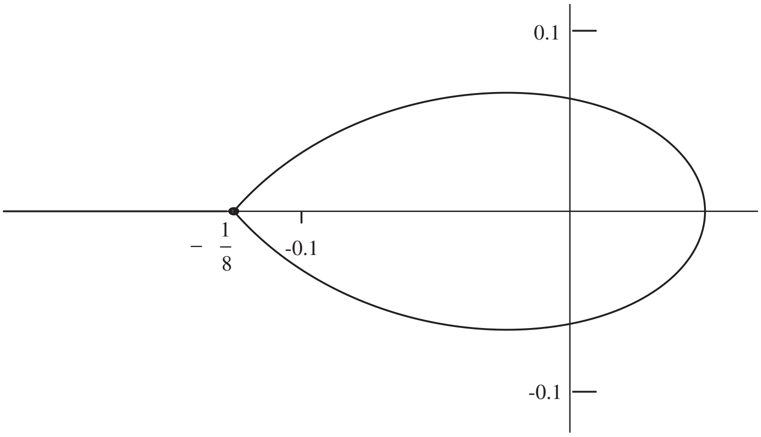

Fig. 1. The curve $|t_1| = |t_2|$ for the basic form $k = 3$.

The zeros of the characteristic polynomial are

$$t_1 = \frac{1 - 20z - 8z^2 + (8z+1)^{3/2}}{2(1-z)^3}, \quad t_2 = \frac{1 - 20z - 8z^2 - (8z+1)^{3/2}}{2(1-z)^3}. \tag{5.7}$$

We write this in the form

$$t_1 = \frac{27 - 18w^2 - w^4 + 8w^3}{16(1-z)^3} = \frac{32(1+w)}{(3+w)^3},$$
$$t_2 = \frac{27 - 18w^2 - w^4 - 8w^3}{16(1-z)^3} = \frac{32(1-w)}{(3-w)^3}, \tag{5.8}$$

where $w = \sqrt{8z+1}$. To find the curve in the w-plane defined by $|t_1| = |t_2|$, we write $w = re^{i\theta}$. This gives the curve described by

$$r = \sqrt{-9 + 6\sqrt{3}\cos\,\theta}, \quad -\tfrac{1}{6}\pi \leqslant \theta \leqslant \tfrac{1}{6}\pi \quad \text{and} \quad \Im w = 0. \tag{5.9}$$

In Fig. 1 we show the curve together with the half-line $z \leqslant -\frac{1}{8}$ in the z-plane. In the domain interior to the curve we have $|t_1| > |t_2|$.

5.3. The domains for basic form $k = 5$

The limits α and β of (4.8) are

$$\alpha = \frac{z-2}{1-z}, \quad \beta = \frac{1}{1-z}. \tag{5.10}$$

The zeros of the characteristic polynomial are

$$t_1 = 1, \quad t_2 = \frac{1}{1-z}. \tag{5.11}$$

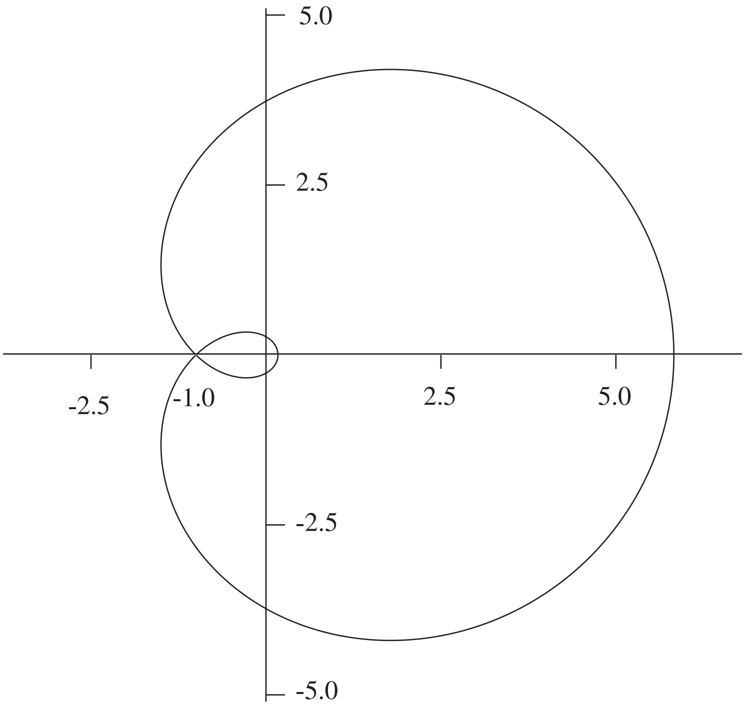

Fig. 2. The curve $|t_1| = |t_2|$ for the basic form $k = 6$.

The equation $|t_1| = |t_2|$ holds when $|1 - z| = 1$, which defines a circle with centre $z = 1$ and radius 1. Inside the circle we have $|t_2| > |t_1|$.

5.4. The domains for basic form $k = 6$

The limits α and β of (4.11) are

$$\alpha = -\frac{z^2 - 6z + 1}{(1 - z)^2}, \quad \beta = -\frac{4z}{(1 - z)^2}. \tag{5.12}$$

The zeros of the characteristic polynomial are

$$t_1 = 1, \quad t_2 = -\frac{4z}{(1 - z)^2}. \tag{5.13}$$

To find the curve defined by $|t_1| = |t_2|$, we write $z = re^{i\theta}$. This gives the curve described by

$$r = 2 + \cos\,\theta \pm \sqrt{\cos^2\theta + 4\cos\,\theta + 3}, \quad -\pi \leqslant \theta \leqslant \pi. \tag{5.14}$$

Both signs give a closed loop with common point -1. In Fig. 2 we show this curve in the z-plane. In the domain interior to the inner curve we have $|t_1| > |t_2|$; between the inner curve and the outer curve we have $|t_1| < |t_2|$, and outside the outer curve $|t_1| > |t_2|$.

5.5. *The domains for basic form $k = 13$*

The limits α and β of (4.14) are

$$\alpha = \frac{1-2z}{z}, \quad \beta = \frac{z-1}{z}. \tag{5.15}$$

The zeros of the characteristic polynomial are

$$t_1 = 1, \quad t_2 = \frac{z-1}{z}. \tag{5.16}$$

The equation $|t_1| = |t_2|$ holds when $\Re z = \frac{1}{2}$. When $\Re z > \frac{1}{2}$ we have $|t_1| > |t_2|$.

In this case we can easily identify pairs of minimal and dominant solutions. From the power series of the Gauss function we find for f_n of (4.16) the estimate $f_n = 1 + \mathcal{O}(1/n)$ for all z. For g_n of (4.16) we apply (2.4), and obtain

$$g_n = \frac{(1-z)^n}{z^n} \frac{z^{1-c}\Gamma(c+n)}{\Gamma(c-a-b+1+n)} {}_2F_1\left({1-b, 1-a \atop c-a-b+1+n}; 1-z \right), \tag{5.17}$$

and again, the Gauss function is $1 + \mathcal{O}(1/n)$ for all z. We conclude that in compact domains of $\Re z < \frac{1}{2}$, f_n is the minimal solution and g_n is a dominant solution. In compact domains of $\Re z > \frac{1}{2}$ the roles of f_n and g_n are interchanged.

6. Asymptotics for minimal and dominant solutions

In some cases we simply use the power series in (1.2), which provides an asymptotic expansion for large c. In some other cases we can use connection formulas for transforming the Gauss function to the case for large c. In a later paper we give details for the non-trivial cases, for example by using results from [8]. This paper gives the asymptotic expansion of any hypergeometric function of type ${}_2F_1(a + \varepsilon_1\lambda, b + \varepsilon_2\lambda; c + \varepsilon_3\lambda; z)$, where $\varepsilon_j = 0, \pm 1$ (with of course all $\varepsilon_j = 0$ excluded) for large complex values of λ and fixed complex a, b, c and z. Watson also treats the form

$${}_2F_1\left({a+\lambda, b+\lambda \atop c+2\lambda}; z \right). \tag{6.1}$$

Certain forms of g_n mentioned in the previous section are of different type, see for example (4.7), and for these cases we need to develop new asymptotic expansions.

7. The cases $k = 15$, $k = 16$, $k = 25$

The cases $k = 15, 16, 25$ in Table 1 are special because we refer to these as ‘change signs in other cases’, and we do not consider them as basic forms. The recursion relations for the cases $k = 15, 16, 25$ are the same as those for $k = 13, 12, 3$, respectively, when we recur backwards, that is to $-\infty$. The zeros t_1 and t_2 of the characteristic polynomial do not change when we change the recursion direction. In our second paper we investigate the cases $k = 15, 16, 25$, separately.

8. Numerical examples

The power series (1.2) is very useful for numerical computations for z properly inside the unit disk. Transformations and connection formulas as in (2.2), (2.3), (3.3), (3.10), (3.15), and (3.16) can be used to cover large parts of the complex z-plane. We see that for the computation of the Gauss function we can use power series with powers of

$$z, \quad 1-z, \quad \frac{1}{z}, \quad \frac{z-1}{z}, \quad \frac{1}{1-z}, \quad \frac{z}{z-1}. \tag{8.1}$$

For numerical computations we need convergence conditions like

$$|z|<\rho, \quad |1-z|<\rho, \quad \left|\frac{1}{z}\right|<\rho, \quad \left|\frac{z-1}{z}\right|<\rho, \quad \left|\frac{1}{1-z}\right|<\rho, \quad \left|\frac{z}{z-1}\right|<\rho, \tag{8.2}$$

with $0<\rho<1$.

Not all points in the z-plane satisfy one of these inequalities for a given number ρ. In Fig. 3 we take $\rho=\frac{3}{4}$. In the dark area at least one of the above inequalities is satisfied. In the light areas, "around" the points $e^{\pm\pi i/3}$, none of these inequalities is satisfied. By choosing ρ closer to unity these light domains become smaller. In [2] power series expansions of the Gauss function are derived, which enable computations near these special points.

For certain combinations of the parameters a, b and c, the connection formulas become numerically unstable. For example, if $c=a+b$, the relation in (3.3) is well-defined, although two gamma functions are infinite. By using a limiting procedure the value of ${}_2F_1(a, b; a+b; z)$ can be found. For c close to $a+b$ numerical instabilities occur when using (3.3). See [3] for many examples and details. Other instabilities in the evaluation of the power series (1.2) may arise for large values of a and b.

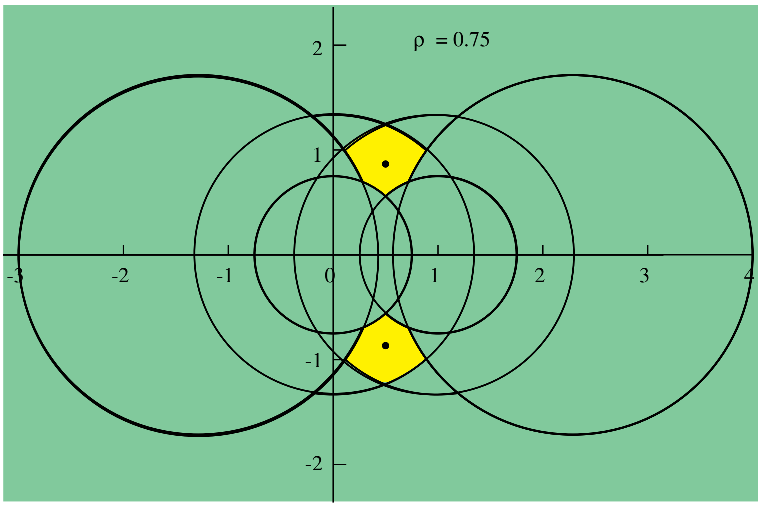

Fig. 3. In the light domains around the points $e^{\pm\pi i/3}$ none of the inequalities of (8.2) is satisfied.

In [9, p. 71] an example is given of how to compute a Gauss function with argument $z = e^{\pi i/3}$, the point that is excluded from the convergence domains shown in Fig. 3. Wimp considers the computation of

$$ {}_2F_1\left({\frac{2}{3}, 1 \atop \frac{4}{3}}; e^{\pi i/3}\right) = \frac{2\pi e^{\pi i/6}\Gamma(\frac{1}{3})}{9[\Gamma(\frac{2}{3})]^2}, \tag{8.3} $$

by using a Miller algorithm for the hypergeometric functions

$$ f_n = {}_2F_1\left({n+a, n+b \atop 2n+c}; z\right), \quad n = 0, 1, 2, \ldots. \tag{8.4} $$

This recursion type was not initially included in the group of 26 discussed in the present paper. However as, discussed in Section 4.1, this recurrence can be related to the case $k = 2$ to conclude that f_n is minimal. Therefore, Miller's algorithm can be applied when a sum rule is provided, as done in [9, p. 71].

We can also use the basic form $k = 13$, with pure c-recursion. From Section 5.5 it follows that for the point $z = e^{\pi i/3} = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$ the solutions of the recursion relation (4.14) are neither dominant nor minimal. We use backward recursion for ${}_2F_1(a, b; c+n; z)$ with two starting values for $n = 29$ and $n = 30$. With these large values of $c+n$ the power series converge fast. In 15D arithmetic we have computed the value of (8.3) with a relative error 2×10^{-14}. The exact value is

$$ 0.883319375142724\ldots + 0.509984679019064\ldots i. $$

With recursion we obtain

$$ 0.883319375142719 + 0.509984679019039i. $$

In [9, p. 72] another example of the Miller algorithm is discussed for the basic form $k = 13$ (pure c-recursion).

Several algorithms based on recursion relations for special cases of the Gauss functions have been published, in particular for computing Legendre functions. For recent papers, see [4–6].

Acknowledgements

A. Gil acknowledges financial support from Ministerio de Ciencia y Tecnología (programa Ramón y Cajal). J. Segura acknowledges financial support from project BFM2003-06335-C03-02. N.M. Temme acknowledges financial support from Ministerio de Educación y Ciencia (Programa de Sabáticos) from project SAB2003-0113.

The authors thank the referees for their valuable comments on the first version of the paper.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, vol. 55, Superintendent of Documents, U.S. Government Printing Office, Washington, DC, 1964, for sale.

[2] W. Bühring, An analytic continuation of the hypergeometric series, SIAM J. Math. Anal. 18 (3) (1987) 884–889.

[3] R.C. Forrey, Computing the hypergeometric function, J. Comput. Phys. 137 (1) (1997) 79–100.

[4] A. Gil, J. Segura, Evaluation of Legendre functions of argument greater than one, Comput. Phys. Comm. 105 (2–3) (1997) 273–283.
[5] A. Gil, J. Segura, N.M. Temme, Computing toroidal functions for wide ranges of the parameters, J. Comput. Phys. 161 (1) (2000) 204–217.
[6] J. Segura, A. Gil, Evaluation of associated Legendre functions off the cut and parabolic cylinder functions, Electron. Trans. Numer. Anal. 9 (1999) 137–146 (electronic). Orthogonal polynomials: numerical and symbolic algorithms, Leganés, 1998.
[7] N.M. Temme, Special Functions. An Introduction to the Classical Functions of Mathematical Physics, Wiley-Interscience, New York, 1996.
[8] G.N. Watson, Asymptotic expansions of hypergeometric functions, Trans. Cambridge Philos. Soc. 22 (1918) 277–308.
[9] J. Wimp, Computation with Recurrence Relations. Applicable Mathematics Series. Pitman Advanced Publishing Program, Boston, MA, 1984.